Short Papers

Stokes Phenomenon in the Development of Microstrip Green's Function and Its Ramifications

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Abstract—**In this paper we examine the effect of truncating an infinite asymptotic series for the Hankel function used in microstrip antenna calculations. In particular, the accuracy of this truncated asymptotic expansion of the Hankel function is examined from a numerical viewpoint. This expansion has been used in the literature to obtain closed-form expressions for the microstrip Green's function for subsequent use in calculating mutual coupling between elements in a microstrip array. In this paper, we show that truncating the asymptotic series for the Hankel function could lead to severe unexpected errors for those values of the argument where the asymptotic expansion is normally expected to be valid. This is known as the Stokes phenomenon and has generally remained obscure in the literature. Since the large argument of the Hankel function is shown to be related to the lateral separation between two antennas, the results presented here have a particular bearing in calculating mutual coupling between widely separated elements in electrically large microstrip arrays.**

I. Introduction

Mutual coupling between microstrip elements usually involves evaluation of Sommerfeld integrals for both source and observer points on the same plane. To facilitate efficient calculation of mutual coupling, closed-form (asymptotic) representations of the Sommerfeld integrals involving Hankel functions were obtained by employing its asymptotic form [1]. Furthermore, it has been reported recently that such formulations can predict and identify many physical effects that were not possible using exact techniques [2]. In [2] it is shown that mutual coupling between widely separated elements decayed quasiperiodically and that it could also become numerically significant at such lateral separations. This implied that accurate quantification of mutual coupling is important in such cases. The formulation in [1] is efficient in calculating mutual coupling between widely separated elements. The subject of this paper is to investigate the limitations of such formulations that have been obtained via approximations.

Large-argument representations of cylinder functions are routinely used for numerical calculations [3]. Emphasizing the effects of probable numerical errors resulting from truncating the infinite asymptotic series for the Hankel function [4], [5] is the major purpose of this paper. Such errors, resulting from truncated asymptotic expansions, are due to Stokes phenomenon [6]–[8]. Unfortunately, this phenomenon does not seem to have been reported or analyzed extensively in the literature. Consequently it appears relevant to illustrate this specific problem with applications to calculating mutual coupling between microstrip antennas.

The scope of the results presented here is general in nature. The conclusions presented are relevant to practical problems involving cylinder functions for large, complex arguments. The present discussion refers to mutual coupling problems in microstrip antennas studied earlier [2], [9]. The Stokes phenomenon can also arise, however, in other situations [10].

Section II illustrates the general features of the Stokes phenomenon in connection with the microstrip antenna problem. Section III contains some numerical results and suggestions for future research into this area. A summary of this work is provided in Section IV.

Manuscript received June 12, 1995; revised August 26, 1996. This work was supported in part by Allied-Signal Targetted University Research program.

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Publisher Item Identifier S 0018-9480(96)08488-8.

II. Analysis of the Problem

It is well known [4] that the microstrip Green's function is expressed in terms of Sommerfeld integrals that contain in their integrands the Hankel function $H_0^{(2)}(z)$. In [1, (59)] a simple residue series form for the microstrip Green's function has also been obtained in terms of $H_0^{(2)}(z)$—the complex argument

$$z = \rho\sqrt{k_0^2 - \beta_p^2}. \tag{1}$$

In (1) β_p is the surface or leaky wave pole location, k_0 is the free-space wavenumber, and ρ is the lateral separation between the two antennas. When $k_0\rho \to \infty, z \to \infty$ in (1), and hence one employs the asymptotic form of $H_0^{(2)}(z)$ [3], [5] to compute the residue series representation of the microstrip Green's function. The mutual impedance is directly proportional to this residue [2] and hence depends on the numerical accuracy of this asymptotic expansion. Truncations of infinite asymptotic expansions lead to numerical inaccuracies as investigated in general in [8]. This aspect is elaborated below.

If an analytic function $f(z)$ of a complex variable z, expressed as a suitably defined contour integral, yields an asymptotic expansion as $z \to z_0$, then it is possible to write the following infinite series [6, p. 21]:

$$f(z) \asymp \sum_{n=0}^{+\infty} c_n \Psi_n \tag{2}$$

valid in some domain Λ in which $z \to z_0$. In (2) z_0 could be a saddle point [4], [6]–[8] that may be close to some other singularity of $f(z)$ like poles or branch points. In (2) $\{\Psi_n\}$ is an asymptotic sequence and $\{c_n\}$ is a sequence of complex coefficients.

For all practical applications, the infinite sum in (2) is generally truncated to a finite number of terms, assuming that the remainder becomes exponentially small as $z \to \infty$. As shown in [8, chs. 21, 22], across certain lines passing through $z = z_0$ this remainder suddenly becomes exponentially large, rendering the finite (or truncated) representation of (2) inaccurate for all practical calculations. This is a manifestation of the Stokes phenomenon and the lines (or rays) through $z = z_0$ are called Stokes lines. It has also been shown that there exists an optimal number of terms, n, for a given value of z [8]. This optimal number yields the best possible truncation of the infinite asymptotic series, so that it is numerically superior to other possible truncations.

To illustrate the effect of truncations on an asymptotic expansion, calculation of cylinder functions is considered. The Stokes phenomenon for cylinder functions is demonstrated by calculation of the Wronskian in the following section.

III. Numerical Results and Discussion

The standard definition of the Hankel function for integer order m and complex argument z is given in the relation [5, p. 358]

$$H_m^{(2)}(z) = J_m(z) - jY_m(z). \tag{3}$$

As suggested in [3], for $|z| \gg m$, the Hankel asymptotic expansions for $J_m(z), Y_m(z)$ can be used. These read from [5, p. 364] as

$$J_m(z) = \sqrt{\frac{2}{\pi z}}[P(m,z)\cos\chi - Q(m,z)\sin\chi] \tag{4}$$

and

$$Y_m(z) = \sqrt{\frac{2}{\pi z}}[P(m,z)\sin\chi + Q(m,z)\cos\chi] \tag{5}$$

where the truncated expressions for the infinite series for $P(m,z), Q(m,z)$ read from [5, p. 364] as

$$P(m,z) \simeq 1 \quad \text{and} \tag{6}$$

$$Q(m,z) \simeq \frac{\eta - 1}{8z}. \tag{7}$$

In (4)–(7) $\chi = z - (m\pi/2 + \pi/4)$ and $\eta = 4 \times m^2$. The accuracy of calculating (3) thus depends on (4) and (5) for $|z| \gg m$. To check the accuracy of (4) and (5) the well-known Wronskian relationship [5, p. 360, (9.1.16)]

$$\begin{aligned} &J_{m+1}(z)Y_m(z) - J_m(z)Y_{m+1}(z) \\ &\quad = J_m(z)Y_m'(z) - J_m'(z)Y_m(z) = \frac{2}{\pi z} \end{aligned} \tag{8}$$

can be used. The asymptotic forms for the derivatives $J_m'(z), Y_m'(z)$ are given in [5] and are

$$J_m'(z) = -\sqrt{\frac{2}{\pi z}}[R(m,z)\sin\chi + S(m,z)\cos\chi] \tag{9}$$

and

$$Y_m'(z) = \sqrt{\frac{2}{\pi z}}[R(m,z)\cos\chi - S(m,z)\sin\chi]. \tag{10}$$

The terms $R(m,z)$ and $S(m,z)$ in (14) and (15) are also truncated from their infinite asymptotic series and read from [5, p. 365, (9.2.15) and (9.2.16)] as

$$R(m,z) \simeq 1 \quad \text{and} \tag{11}$$

$$S(m,z) \simeq \frac{\eta + 3}{8z}. \tag{12}$$

Equations (4)–(12) are valid for $|z| \to \infty$ and hence are formally asymptotic [6]. The accuracy of the truncation of $H_0^{(2)}(z)$ for $z \to \infty$ was computed from the Wronskian check based on (8). Truncation type 1 corresponds to results obtained via (6), (7), (11), and (12), and the results shown here correspond to this specific truncation.

The relative error resulting in calculating the Wronskian in (8) was determined for various values of z and integer orders of m. This error (in %) is determined by

$$\epsilon = \frac{|\mathcal{W}_a - \mathcal{W}_e|}{|\mathcal{W}_e|} \times 100 \tag{13}$$

where $\mathcal{W}_a$ is the Wronskian based on (4)–(10) and $\mathcal{W}_e = 2/\pi z$ is its exact value. The results are shown in Figs. 1 and 2. In these figures $z = |z|e^{j\theta}$ where θ is the phase of the complex argument in degrees. Equation (13) provides a simple yet rigorous check for demonstrating the Stokes phenomenon arising due to truncations in P, Q, R, and S for $z \to \infty$. All the computations were done using double-precision complex arithmetic in FORTRAN-77.

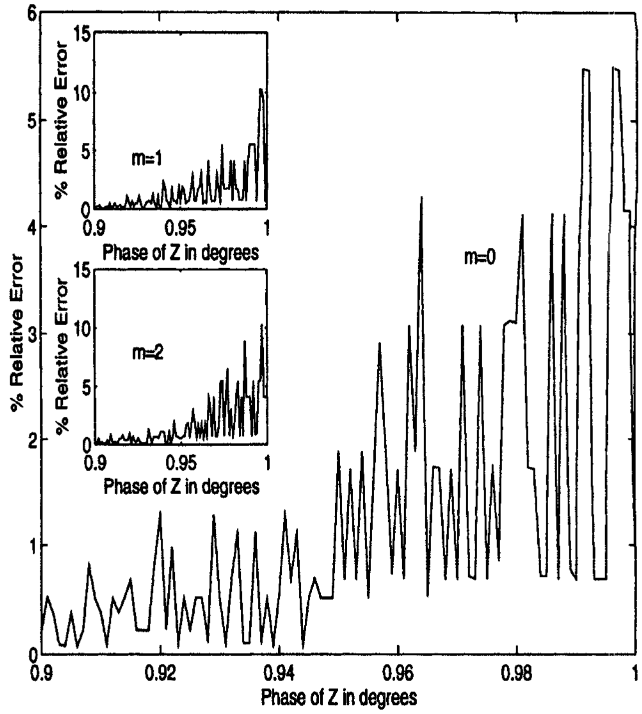

Fig. 1. Comparison of relative errors in the Wronskian at $|z| = 1000$ and for different orders of $m = 0, 1$, and 2.

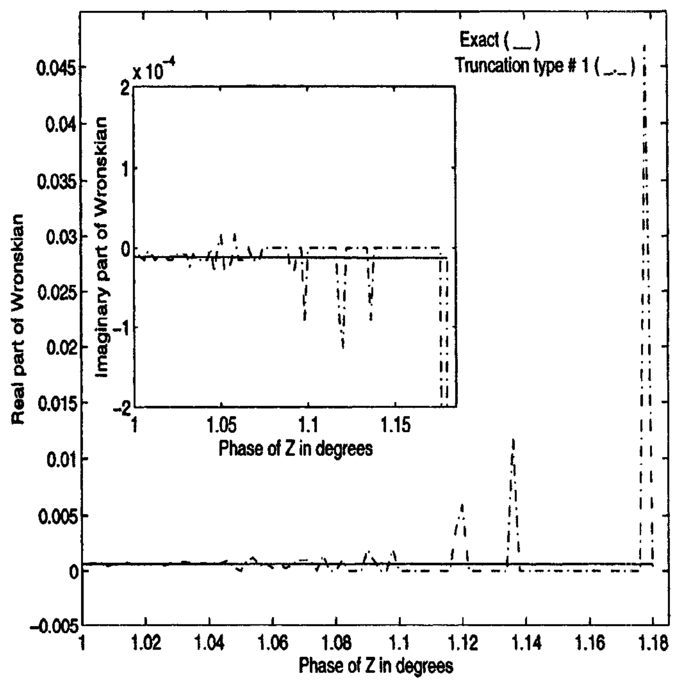

Fig. 2. Comparison of relative errors in the Wronskian at $|z| = 1000$ and for $m = 0$.

In Fig. 1 for all cases of $m = 0, 1$, and 2 and $|z| = 1000$, the relative error increases with increasing θ. The data shown here indicate that the errors could be severe for higher orders. For instance, at $\theta \simeq 1^\circ$, the errors are about 10% for $m = 1$ and 2, while at $m = 0$ they are about 5%. As θ increases, z rotates in the complex plane for a fixed value $|z| = 1000$. This rotation causes z to approach the Stokes lines, which are near $\theta = 1^o$. It is seen that for all orders the relative errors are maximum near this region. In Fig. 2, comparisons between truncation 1 and exact results are shown. The Wronskian computations were done for $m = 0$, corresponding to $\eta = 0$. The results clearly indicate that the truncation is subject to errors.

For commonly used practical microstrip configurations, the relation

$$L = k_0 d\sqrt{\epsilon_r - 1} \leq \frac{\pi}{2} \tag{14}$$

is well known [2]. Here d is the substrate thickness, ϵ_r is the relative permittivity, and L is the electrical length. This will excite one TM surface- and one TE leaky-wave pole [1]. For $\epsilon_r = 4$, we find from [9, Fig. 4] that $\beta_p/k_0 \simeq 2.7 - j8.0$. Substituting these values in (1) we get

$$z \simeq 53.35\frac{\rho}{\lambda}e^{+j1.89}. \tag{15}$$

As shown in [2, Figs. 1, 2, 5, 6], lateral separations of $\rho \geq 20\lambda$ are not uncommon in designing large arrays. Setting $\rho = 20\lambda$ in (18) gives $|z| \simeq 1067$. One can conclude from Figs. 1 and 2 that truncations in the asymptotic series for $H_0^{(2)}(z)$, for $|z| \geq 1000$, can be subject to increased numerical errors.

Our results indicate that the Stokes phenomenon could eventually dictate the accuracy of computing the mutual coupling for medium or large microstrip arrays. Techniques such as the Borel summation formula [8, pp. 405–408] appear applicable although much work remains to be done in the future.

IV. Summary

In this paper we have studied the effects of truncations of the infinite asymptotic series for the Hankel function that appears in the Sommerfeld integral for the microstrip Green's function. For large values of the complex argument z, such truncated expansions can be inaccurate. This inaccuracy is a manifestation of the Stokes phenomenon that depends both on the magnitude and phase of the complex argument z, which depends on the substrate geometry and the lateral separation between antennas. When z tends to a transition (or distinguished) point z_0, certain rays in the complex z plane are crossed, across which the truncated asymptotic expansion is no longer analytically continuable; these are called Stokes lines. This leads to numerical inaccuracies that may manifest themselves in calculating mutual coupling between widely separated elements in a microstrip array. It has been found numerically that for $|z| \geq 1000$ the Stokes phenomenon manifests itself when the Green's function is computed; hence, the mutual coupling between microstrip antennas. This value generally corresponds to the dimensions of a medium-sized array for electrically thin substrates with relatively low permittivities. To rectify the Stokes phenomenon the Borel summation formula may be used, but its application to the asymptotic evaluation of the Sommerfeld integral remains a challenging topic for future research.

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A Fast Algorithm for Computing Field Radiated by an Insulated Dipole Antenna in Dissipative Medium

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***Abstract*—A fast algorithm for determining the near-field characteristics of an insulated dipole antenna (IDA) embedded in a homogenous dissipative medium is described in this paper. A thin-wire-approximation type of analysis is followed here. In this case, radiation is considered to originate from a filamentary current flowing along the axis of the dipole, which is surrounded immediately by the ambient dissipative medium. The translational symmetry inherent in the resultant radiation integrals is then exploited to speed up the computation. In one case studied, the basic thin-wire approach that uses no symmetry property is found to yield accurate results in approximately 380 times less CPU time than the traditional King–Casey approach. In another case, use of symmetry property further reduces the CPU time by a factor of 7; additional reduction in CPU time is possible by taking into account the near-field nature of the problem.**

I. Introduction

Analysis of the near field characteristics of an insulated dipole antenna (IDA) is fundamental in the design and evaluation of the heating performance of an interstitial microwave hyperthermia system. For the field computation purpose, IDA's may be classified as being either uniformly or nonuniformly insulated. In this paper, a fast computing algorithm will be developed explicitly for the uniformly insulated IDA's shown in Fig. 1, and extension to the nonuniformly insulated IDA's will also be described.

Two types of analysis have been employed in the past. In the King-Casey analysis of the symmetrically fed, uniformly insulated IDA shown in Fig. 1(a) [1], [2], the IDA is first treated as a lossy transmission line while determining the antenna input impedance and equivalent electric and magnetic current sources present over the exterior surface of the insulating catheter. The latter are then used

Manuscript received November 11, 1995; revised August 26, 1996. This work was supported by the National Science Council of the Republic of China under Grants NSC 83-0420-E-009-002 and NSC 84-2213-E-009-052.

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Publisher Item Identifier S 0018-9480(96)08490-6.